Logarithms and Venn diagrams

Think about all positive integers a and b and the three properties:

A: $\log_{10} a - \log_{10} b$ is an integer

B: $\log_{10} a \times \log_{10} b < 0$

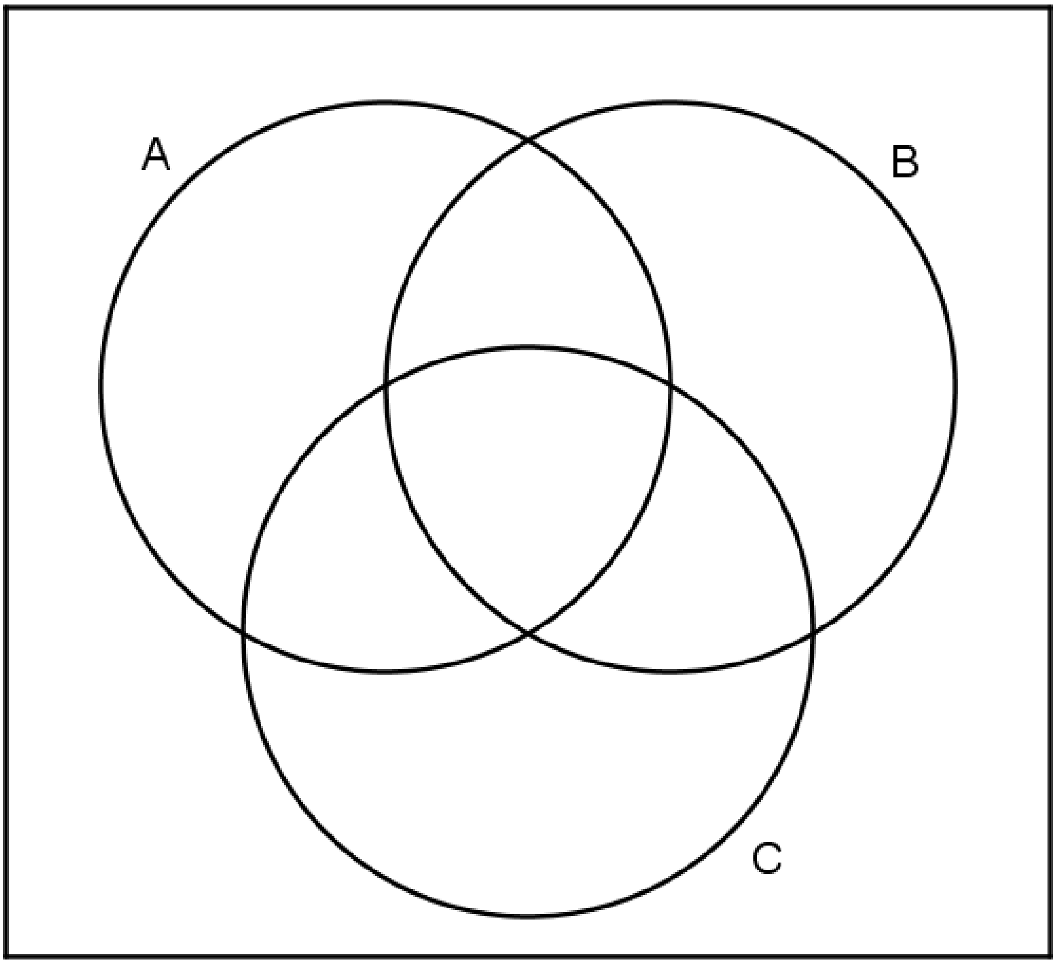
C: $\log_{10} a + \log_{10} b$ is an integer

Can you find values of a and b which satisfy all three properties A, B and C?
If so write them in the central region where the three circles overlap.

How about values which don't satisfy any of the requirements?
Fill this in the region outside the three circles.

The task: To find one example for each region.

Is it possible to find an example for every region?



Logarithms Questions

1. In the news on February 7, 2013, it was announced that a new prime number had been discovered which had 17,425,170 digits. It is of the form $2^p - 1$ where p is a prime number.
How might you find the value of p?

2. (OCR 2000 Paper STEP Mathematics 1 Question 1)

To nine decimal places, $\log_{10} 2 = 0.301029996$ and $\log_{10} 3 = 0.477121255$.

(i) Calculate $\log_{10} 5$ and $\log_{10} 6$ to three decimal places. By taking logs, or otherwise, show that $5\times10^{47} < 3^{100} < 6\times10^{47}$
Hence write down the first digit of 3^{100}.

(ii) Find the first digit of each of the following numbers: $2^{1000}; 2^{10000};$ and 2^{100000}.

3. Which positive integers can be expressed using three 2s and any mathematical operations you wish?

Here are some examples: $4=\sqrt{2^{2\times2}}$, $5=\dfrac{2}{.2\times2}$, $6={}^{2+2}C_2$, $7=\dfrac{2}{.\dot{2}}-2$

4. Evaluate the sum of the logarithms to base 10 of all the factors of 1000000.

5. Use the facts that $3^2 > 2^3$ and $3^5 < 2^8$ to find upper and lower bounds for the value of $\log_2 3$.

6. Prove that $\log_2 3$ is irrational.

7. Find the sum of the first 100 terms in each of the following series

$S=\log_{10}3+\left(\log_{10}3\right)^2+\left(\log_{10}3\right)^3+\ldots$ $\quad$ $T=\log_{10}3+\log_{10}\left(3^2\right)+\log_{10}\left(3^3\right)+\ldots$

8. The spreadsheet illustrates that $\log\left(a^2\right)=2\log a$ for a list of randomly generated values of a.

A2 =RAND()*10

	A	B	C
	a	log(a^2)	2log(a)
	8.14386	1.82166	1.82166
	1.36033	0.26729	0.26729
	3.4563	1.07722	1.07722
	9.87441	1.98902	1.98902
	0.65898	-0.36226	-0.36226

After hiding the column showing the value of a, use the value of $\log\left(a^2\right)$ or $2\log a$ to estimate the hidden value of a without a calculator. You might want to choose one carefully.

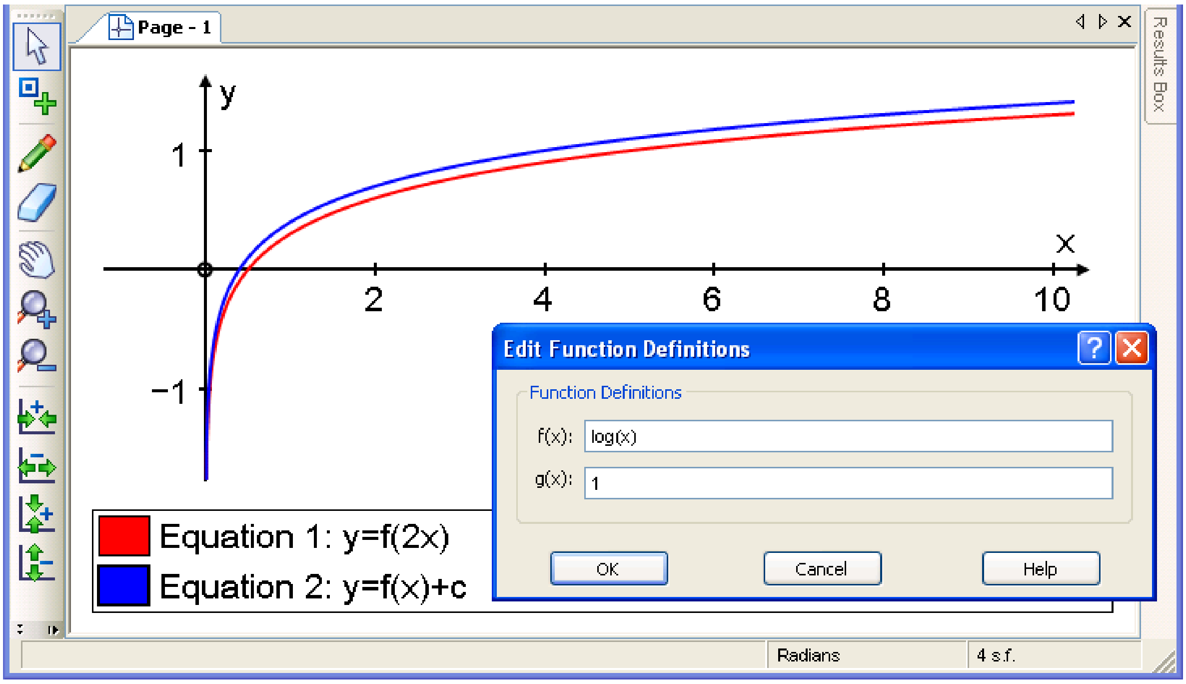
	B	C
1	log(a^2)	2log(a)
2	1.57643	1.57643
3	1.50157	1.50157
4	1.37383	1.37383
5	1.89003	1.89003
6	1.62797	1.62797

For example,

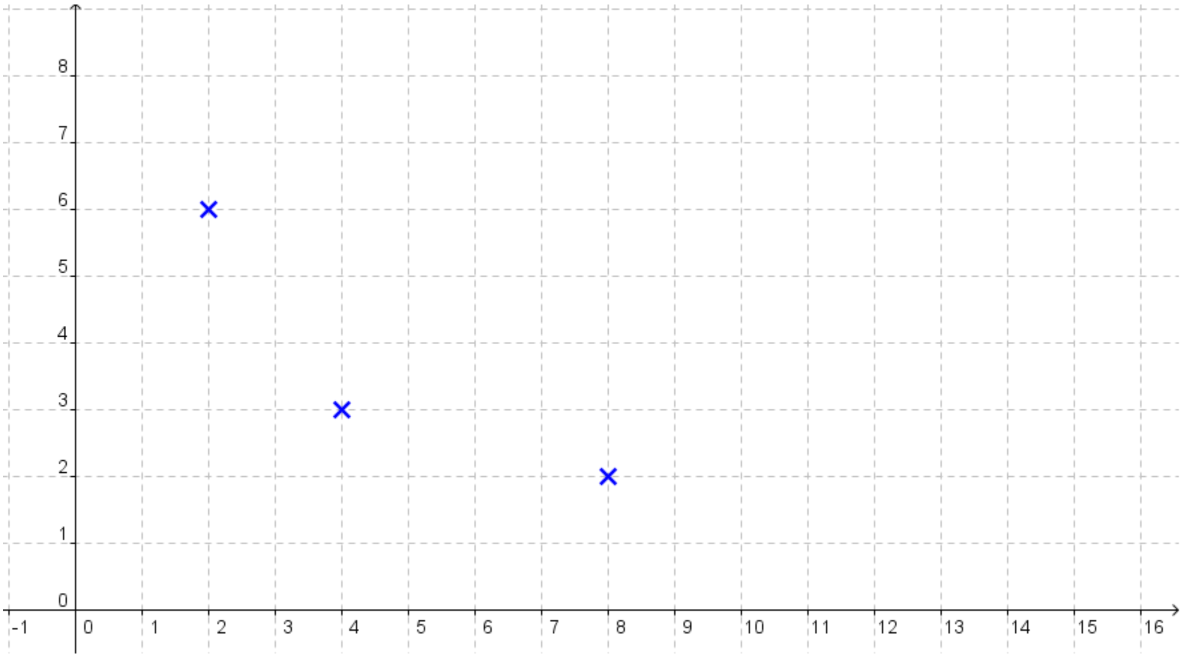
$$2\log a = 1.37 \quad\Rightarrow\quad \log a \approx \frac{2}{3} \quad\Rightarrow\quad a \approx 10^{\frac{2}{3}} = \sqrt[3]{100}$$

4.5 is a bit small since $\left(\frac{9}{2}\right)^3 = \frac{729}{8} \approx 91$ so estimate $a \approx 4.6$. You can reveal the column to check.

9. For $\mathrm{f}(x)=\log_{10}x$, $y=\mathrm{f}(2x)$ and $y=\mathrm{f}(x)+c$ turn out to be exactly the same for one value of c. What is this value?



10. Think about values of x and y satisfying $x^y = 64$. Three are shown on the graph below.
What does the complete graph of $x^y = 64$ look like?
How can logarithms give you an insight into this graph?



11. A bank offers savers a rate of 4% compound interest. For how long would a saver need to leave money in such an account in order for it to double in value?
In general, if a bank offered a rate of x%, how long would it take for an amount to double?

12. Simplify $\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \frac{1}{\log_4 N} + \ldots + \frac{1}{\log_{100} N}$

13. Evaluate, without using a calculator, $9^{\log_3 4}$. Make up other expressions like this.

Solutions

1. Since $2^p - 1$ has 17,425,170 digits, then $10^{17,425,169} < 2^p < 10^{17,425,170}$.
Taking logs to base 10 this becomes $17,425,169 < p\log_{10} 2 < 17,425,170$.
(Notice that we have used the fact that $\log_{10} x$ is an increasing function here.)
Dividing by $\log_{10} 2$ gives $57,885,159 \le p \le 57,885,161$. Since 57,885,159 is a multiple of 3 and 57,885,160 is even, the answer must be 57,885,161 which is indeed prime.
Since $x^n - 1$ has a factor of $x - 1$ by the factor theorem, $2^{ab} - 1 = \left(2^a\right)^b - 1$ is divisible by $2^a - 1$. Therefore, for a number of the form $2^n - 1$ to be prime, it is a necessary (but not sufficient) condition that n is prime.

2 (i) $\log_{10} 5 = \log_{10}\left(\frac{10}{2}\right) = \log_{10} 10 - \log_{10} 2 = 1 - 0.301029996 = 0.699 \text{ (3d.p.)}$
$\log_{10} 6 = \log_{10} 2 + \log_{10} 3 = 0.301 + 0.477 = 0.778 \text{ (3d.p.)}$

$\log_{10}\left(5\times10^{47}\right) = \log_{10} 5 + 47 = 47.699$
$\log_{10}\left(3^{100}\right) = 100\log_{10} 3 = 100\times0.47712 = 47.712$
$\log_{10}\left(6\times10^{47}\right) = \log_{10} 6 + 47 = 47.778$
Since $\log_{10}\left(5\times10^{47}\right) < \log_{10}\left(3^{100}\right) < \log_{10}\left(6\times10^{47}\right)$ then $5\times10^{47} < 3^{100} < 6\times10^{47}$
So the first digit of 3^{100} is 5.

(ii) $\log_{10} 2^{1000} = 1000\log_{10} 2 = 1000\times0.30103 = 301.03$

$\log_{10} 10^{300} = 300$ $\quad$ $\log_{10} 2\times10^{300} = \log_{10} 2 + 300 = 300.301$

$\therefore 10^{300} < 2^{1000} < 2\times10^{300}$ and so the first digit is 1

$\log_{10} 2^{10000} = 10000\log_{10} 2 = 10000\times0.301029996 = 3010.29996$

$\log_{10} 10^{3010} = 3010$ $\quad$ $\log_{10} 2\times10^{3010} = \log_{10} 2 + 3010 = 3010.301$

$\therefore 10^{3010} < 2^{10000} < 2\times10^{3010}$ and so the first digit is 1

$\log_{10} 2^{100000} = 100000\log_{10} 2 = 100000\times0.301029996 = 30102.9996$

$\log_{10} 9\times10^{30102} = 30102 + \log_{10} 9 = 30102 + 2\log_{10} 3 = 30102.95424$ and $\log_{10} 10^{30103} = 30103$

$\therefore 9\times10^{30102} < 2^{100000} < 10^{30103}$ and so the first digit is 9

(Please note. The solution is not produced by OCR.)

3. Surprisingly, any positive integer can be expressed using nested square roots as follows:

$$-\log_2\left(\log_2\sqrt{2}\right)=-\log_2\left(\log_2 2^{\frac{1}{2}}\right)=-\log_2\left(\frac{1}{2}\right)=-(-1)=1$$

$$-\log_2\left(\log_2\sqrt{\sqrt{2}}\right)=-\log_2\left(\log_2 2^{\frac{1}{2^2}}\right)=-\log_2\left(\frac{1}{2^2}\right)=-(-2)=2$$

$$-\log_2\left(\log_2\sqrt{\sqrt{\sqrt{2}}}\right)=-\log_2\left(\log_2 2^{\frac{1}{2^3}}\right)=-\log_2\left(\frac{1}{2^3}\right)=-(-3)=3$$

In general, $N=-\log_2\left(\log_2 x\right)$ where x is the number obtained by taking N successive square roots of 2.

4. Every factor is of the form 2^m5^n where $0\le m,n\le 6$.

We have 7 choices for each of m and n and so there are $7\times 7=49$ factors which occur in 24 pairs, each of whose product is 1000000, along with 1000, the square root, which stands alone.

For example, 50 pairs off with 20000 and 160 pairs off with 6250.

$$\sum_{a|1000000}\log a=\sum_{a|1000000,a<1000}\left(\log a+\log\frac{1000000}{a}\right)+\log 1000$$

$$\begin{aligned}&=24\times\log 1000000+3\\&=24\times 6+3\\&=147\end{aligned}$$

5. Take logarithms to base 2 in both inequalities:

$$\log_2 3^2>\log_2 2^3 \quad\Rightarrow\quad 2\log_2 3>3 \quad\Rightarrow\quad \log_2 3>1.5$$

$$\log_2 3^5<\log_2 2^8 \quad\Rightarrow\quad 5\log_2 3<8 \quad\Rightarrow\quad \log_2 3<1.6$$

Therefore $1.5<\log_2 3<1.6$ (Actually $\log_2 3\approx 1.585$)

Similarly, taking logarithms to base 3 of both inequalities leads to $\frac{5}{8}<\log_3 2<\frac{2}{3}$

6. $\log_2 3$ is the solution of the equation $2^x = 3$.

Assume x is rational, letting $x = \frac{p}{q}$ where p and q are integers.

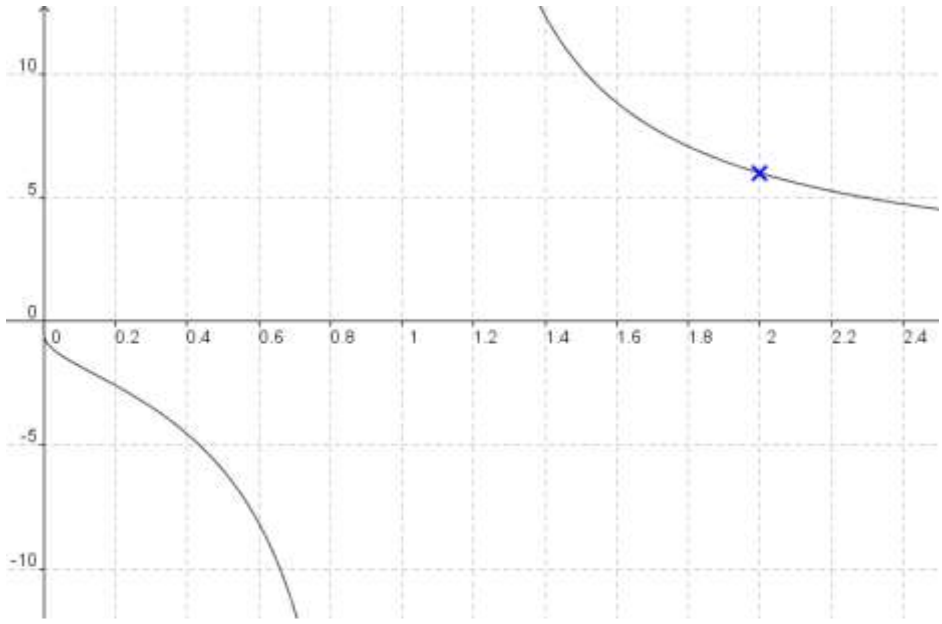
Then $2^x = 3$ becomes $2^p = 3^q$. But by unique prime factorisation no (non-zero) power of 2 can also be a power of 3. Therefore our initial assumption, that x was rational, leads to a contradiction. So x must be irrational.

7 $$S = \log 3 + (\log 3)^2 + (\log 3)^3 + \ldots + (\log 3)^{100} = \frac{\log 3\left(1-(\log 3)^{100}\right)}{1-\log 3} \approx \frac{\log 3}{1-\log 3}$$

$$T = \log 3 + \log\left(3^2\right) + \log\left(3^3\right) + \ldots + \log\left(3^{100}\right) = \log 3(1+2+3+\ldots+100) = 5050\log 3$$

9. $c = \log_{10} 2$

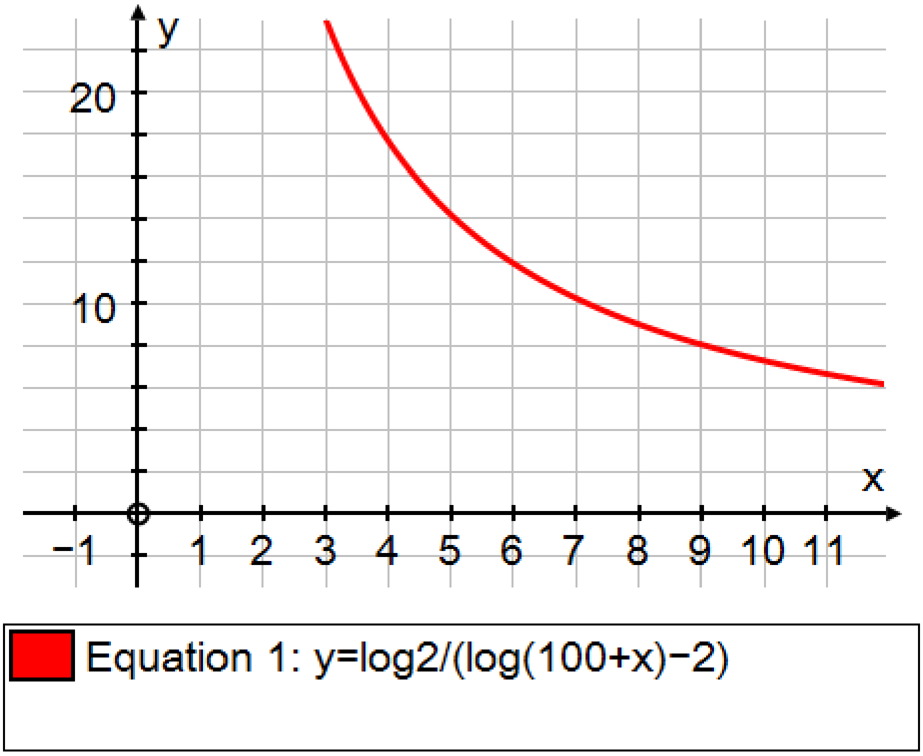
10. The interesting region is $0 < x < 1$. $x^y = 64$ is the same as $y = \frac{\log 64}{\log x}$ so thinking about the graph of $y = \log x$ and then $y = \frac{1}{\log x}$ helps to understand the shape.



11. We need to solve $2 = 1.04^n$. This gives $n = \frac{\log 2}{\log 1.04} \approx 17.7$ years

In general, the number of years, y, is given by $y = \frac{\log 2}{\log\left(1 + \frac{x}{100}\right)}$ or, equivalently,

$y = \frac{\log 2}{\log(100 + x) - 2}$. The graph is shown below.



12. If $N = a^m$ then $\log_a N = m$ and $\log_N a = \log_N N^{\frac{1}{m}} = \frac{1}{m}$.

This shows that $\frac{1}{\log_a N} = \log_N a$. Therefore

$$\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \frac{1}{\log_4 N} + \ldots + \frac{1}{\log_{100} N} = \log_N 2 + \log_N 3 + \log_N 4 + \ldots + \log_N 100$$

$$= \log_N (2 \times 3 \times 4 \ldots \times 100)$$

$$= \log_N 100!$$

$$= \frac{1}{\log_{100!} N}$$

13. $\log_3 4 = x \quad \Rightarrow \quad 3^x = 4$

So $9^{\log_3 4} = 9^x = \left(3^2\right)^x = \left(3^x\right)^2 = 4^2 = 16$

Questions to promote deeper mathematical thinking.

1. Is taking logs a useful technique for solving the equation $3^x = x^3$?

2. Why does the calculator report an error when asked for $\log_{10}(-100)$?

3. Why is the graph of $y = 10^x$ the reflection of the graph $y = \log_{10} x$ in the line $y = x$?

4. Which is bigger $(\log_{10} 1001 - \log_{10} 1000)$ or $(\log_{10} 11 - \log_{10} 10)$?
What does this tell you about the graph of $y = \log_{10} x$?

5. Make up three questions that show you understand the laws of logarithms.

6. Tell me three ways of solving the equation $3^{2x} = 2 \times 5^x$.

7. How would you explain the rule $\log(ab) = \log a + \log b$?

8. What is the same and what is different about indices and logarithms?

9. Explain connections between the law $\log(ab) = \log a + \log b$ and the rule $10^a \times 10^b = 10^{a+b}$.

10. How can we be sure that the graph of $y = \log_{10} x$ does not have a horizontal asymptote?

11. Is it always, sometimes, or never true that $\log_a b \times \log_b a = 1$?

12. Why do $2(\log_{10} 15 - 1)$, $\log_{10} 9 + \log_{10} \frac{1}{4}$, $2\log_{10} 3 - \log_{10} 4$ all have the same value?